

# BIJECTIVITY AND TRAPPING REGIONS FOR COMPLEX CONTINUED FRACTION TRANSFORMATION

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**ABSTRACT.** This paper provides some preliminary results on the dynamics of certain complex continued fractions. After establishing some general number theoretic results, we explore the dynamics of the natural extension map associated to a specific complex continued fraction algorithm (the “diamond” algorithm). We prove that this map has a bijectivity domain that is a subset of a trapping region for the map and, moreover, that both these sets have a “finite product structure” arising from a finite partition specific to the particular algorithm.

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## 1. INTRODUCTION

Number theoretic properties of complex continued fractions were studied classically by Hurwitz [5] and Khinchin [12] and more recently by Doug Hensley [4] and S. G. Dani and Arnaldo Nogueira [3]. Dynamical properties of real continued fractions, namely their connection to coding geodesics on the modular surface, go back to Artin [2], with further development by Caroline Series [13] and Adler and Flatto [1]. Katok and Ugarcovici [9, 10, 11] describe a two-parameter family of minus continued fraction algorithms, which they call “ $(a, b)$ -continued

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fractions.” They describe some number theoretic properties, the dynamics of the associated natural extension maps, and applications to coding geodesics.

The main result of [10] is that, with a few exceptions, the natural extension map  $F_{a,b}$  on  $\mathbb{R}^2$  has a global attractor set consisting of two connected components with “finite rectangular structure,” i.e., bounded by non-decreasing step functions with a finite number of steps. The goal in this paper is to reach a similar result for the natural extension map associated to a particular complex continued fraction algorithm.

Section 2 relates several properties of minus complex continued fractions, most of which are clear analogues of results in [3] for plus complex continued fractions. Section 3 gives a definition of “finite product structure” for sets in  $\mathbb{C}^2$  and discusses the “diamond algorithm” on which the remainder of the paper is focused. The two main results in Sections 4 and 5 are the existence and explicit description of a bijectivity domain  $D_\diamond$  (Theorem 7) a trapping region  $\Psi_\diamond \supset D_\diamond$  (Theorem 8), both of which have a finite product structure. The proof that  $\Psi_\diamond$  traps points (Lemma 9) depends on complex continued fraction theory, namely Theorem 4.

## 2. COMPLEX CONTINUED FRACTIONS

A *continued fraction*, or *c.f.* for short, is any expression of the form

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{\ddots + \frac{b_n}{a_n}}}} \quad \text{or} \quad a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{\ddots}}}$$

A *minus continued fraction* is one in which  $b_n = -1$  for all  $n$ , and a *plus continued fraction* has all  $b_n = +1$ . For the most part, this paper will deal only with infinite minus continued fractions.

Given the two sequences  $\{a_n\}$  and  $\{b_n\}$ , one can define sequences  $\{p_n\}$  and  $\{q_n\}$  by

$$\begin{array}{lll} p_{-2} = 0 & p_{-1} = 1 & p_n = a_n p_{n-1} + b_n p_{n-2} \quad \text{for } n \geq 0; \\ q_{-2} = -1 & q_{-1} = 0 & q_n = a_n q_{n-1} + b_n q_{n-2} \quad \text{for } n \geq 0. \end{array}$$

Algebraic manipulations show that for all  $n \geq 0$ ,

$$\frac{p_n}{q_n} = a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{\ddots + \frac{b_n}{a_n}}}}$$

assuming  $a_n \neq 0$ . This holds for  $a_k$  and  $b_k$  in any ring or field, not necessarily  $\mathbb{R}$  or  $\mathbb{C}$ . The fraction  $\frac{p_n}{q_n}$  is called the  $n^{\text{th}}$  *convergent* of the continued fraction.

Since we deal only with minus continued fractions from now on, we introduce the notations

$$[a_0, a_1, a_2, \dots, a_n] = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{\ddots - \frac{1}{a_n}}}}$$

and

$$[a_0, a_1, a_2, \dots] = a_0 - \frac{1}{a_1 - \frac{1}{a_2 - \frac{1}{\ddots}}}$$

Occasionally in proofs, these notations may be used with non-integer  $a_n$ , and the notation  $[a_0, a_1, a_2, \dots]$  may be used as a formal expression even if the sequence  $r_n = [a_0, a_1, \dots, a_n]$  has no limit as  $n \rightarrow \infty$ .

We also simplify the recurrence relations above to

$$\begin{array}{lll} p_{-2} = 0 & p_{-1} = 1 & p_n = a_n p_{n-1} - p_{n-2} \quad \text{for } n \geq 0; \\ q_{-2} = -1 & q_{-1} = 0 & q_n = a_n q_{n-1} - q_{n-2} \quad \text{for } n \geq 0. \end{array} \quad (1)$$

Denote the set of Gaussian integers by

$$\mathcal{G} = \{x + yi : x, y \in \mathbb{Z}\}.$$

For the remainder of this paper, elements of  $\mathcal{G}$  may also be referred to as complex integers or simply integers. Additionally, a *rational* complex number is an element of  $\mathbb{Q}(i)$ , and therefore an *irrational* complex number is one for which the real or imaginary parts or both are irrational.

Plus complex continued fractions have been studied by Adolf Hurwitz [5], Doug Hensley [4], and more recently by S.G. Dani and Arnaldo Nogueira [3], who introduce the terms “choice function” and “iteration sequence.”<sup>1</sup> A *choice function* is a function  $c : \mathbb{C} \rightarrow \mathcal{G}$  for which  $c(0) = 0$  and  $|z - c(z)| \leq 1$  for all  $z$  (that is,  $c(z)$  chooses a Gaussian integer that is at most a distance of 1 from  $z$ ). For any given choice function, one can define the set

$$\Phi_c = \overline{\{z - c(z) : z \in \mathbb{C}\}}.$$

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<sup>1</sup>The sequence  $\{z_n\}$  defined in (2) is an example of an iteration sequence. Dani and Nogueira give conditions for an arbitrary sequence  $\{z_n\}$  to be useful for constructing continued fractions, but this paper does not deal with such general iteration sequences.

The most classical example is the *Hurwitz* or *nearest integer* choice function which maps each complex number to its nearest Gaussian integer (with some convention for points equidistant from multiple nearest Gaussian integers). This algorithm was discussed in detail by Hurwitz [5], and in this case the set

$$\Phi_{\text{Hurwitz}} = \left\{ x + yi : |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2} \right\}$$

is a unit square centered at the origin.

**Remark.** The definition of a choice function implies that  $\Phi_c \subset \overline{B(0, 1)}$ . In many cases, such as Hurwitz, it is also true that  $\Phi_c \subset B(0, 1)$  or even that  $\Phi_c \subset B(0, r)$  for some  $r < 1$ . There are some number theoretic results that require this additional restriction on  $c$ , but many do not.

One can also construct a choice function starting with a valid set  $\Phi$ .

**Lemma 1.** *Let  $\Phi \subset \overline{B(0, 1)}$  contain 0 and let*

$$f_\emptyset(z) = \begin{cases} z - 1 & \text{if } -\pi/4 \leq \arg z < \pi/4 \\ z - i & \text{if } \pi/4 \leq \arg z < 3\pi/4 \\ z + 1 & \text{if } 3\pi/4 \leq \arg z \text{ or } \arg z < -3\pi/4 \\ z + i & \text{if } -3\pi/4 \leq \arg z < -\pi/4. \end{cases}$$

*If for any  $z \notin \Phi$  there exists an integer  $N_z \geq 0$  such that  $f_\emptyset^{N_z}(z) \in \Phi$ , then the function*

$$c_\Phi(z) = \begin{cases} 0 & \text{if } z \in \Phi \\ f_\emptyset^{N_z}(z) & \text{if } z \notin \Phi \end{cases}$$

*is a valid choice function.*

**Remark.** The two processes of moving from  $c$  to  $\Phi$  and from  $\Phi$  to  $c$  are not inverses: for a choice function  $h$ , the choice function  $c(\Phi_h)$  may not be equal to  $h$ .

For a given choice function  $c$ , sequences  $\{z_n\}$  and  $\{a_n\}$  are defined by

$$z_0 = z, \quad a_n = c(z_n), \quad z_{n+1} = \frac{-1}{z_n - a_n} \quad \text{for } n \geq 0. \quad (2)$$

Dani and Noguiera [3] deal exclusively with plus continued fractions, but the relevant statements can be re-stated and re-proved for minus c.f. For example, a version of Lemma 2 below is stated for plus c.f. as part of Proposition 3.3 in [3], where it has an additional  $(-1)^{n+1}$  term on the right-hand side corresponding to the fact that  $p_{n+1}q_n - q_{n+1}p_n = (-1)^n$  for plus c.f. but equals 1 for all  $n \geq 0$  for minus c.f.

**Lemma 2.** *Let  $\{a_n\}$  and  $\{z_n\}$  be sequences satisfying  $z_0 = z$  and  $z_{n+1} = \frac{-1}{z_n - a_n}$ . Define  $\{p_n\}$  and  $\{q_n\}$  by (1). Then*

$$p_n - q_n z = (z_1 \cdots z_{n+1})^{-1}$$

for all  $n \geq 0$ .

*Proof by induction.* For  $n = 0$ , we have  $p_0 = a_0$ ,  $q_0 = 1$ , and  $z_1 = \frac{-1}{z_0 - a_0}$ , so by direct calculation  $p_0 - q_0 z = z_1^{-1}$ . Now let  $n \geq 1$  and assume  $p_k - q_k z = (z_1 \cdots z_{k+1})^{-1}$  for  $k = 0, \dots, n-1$ . Then

$$\begin{aligned} p_n - q_n z &= (a_n p_{n-1} - p_{n-2}) - (a_n q_{n-1} - q_{n-2})z \\ &= a_n p_{n-1} - p_{n-2} - a_n q_{n-1} z + q_{n-2} z \\ &= a_n (p_{n-1} - q_{n-1} z) - (q_{n-2} z - p_{n-2}) \\ &= a_n (z_1 \cdots z_n)^{-1} - (z_1 \cdots z_{n-1})^{-1} \\ &= (z_1 \cdots z_n)^{-1} (a_n - z_n) \\ &= (z_1 \cdots z_n)^{-1} z_{n+1}^{-1} \\ &= (z_1 \cdots z_{n+1})^{-1} \end{aligned} \quad \square$$

**Lemma 3.** *Under the setup of Lemma 2,  $|z_1 \cdots z_n| \rightarrow \infty$  as  $n \rightarrow \infty$ .*

See [3, Prop. 3.6] for a proof for plus continued fractions; this proof applies equally well to minus c.f. when “ $z_{n+1} = \frac{1}{\beta_n + \zeta_n}$ ” is replaced by “ $-z_{n+1} = \frac{1}{\beta_n + \zeta_n}$ ”.

**Theorem 4.** *Let  $c : \mathbb{C} \rightarrow \mathcal{G}$  be a choice function such that  $|z - c(z)| \neq 1$  if  $z$  is irrational and  $|z| = 1$ . Let  $z \in \mathbb{C}$  be irrational, let  $\{a_n\}$  be the sequence of defined by (2) above, and let  $\{p_n\}$  and  $\{q_n\}$  be defined exactly as in (1). Then  $q_n \neq 0$  for all  $n \geq 0$ , the sequence  $\{\frac{p_n}{q_n}\}$  converges to  $z$ , and  $|q_n| \rightarrow \infty$  as  $n \rightarrow \infty$ .*

*Proof.* Note that  $|z_n| \geq 1$  for all  $n \geq 1$  because  $|z_n - c(z_n)| \leq 1$  by the definition of a choice function and  $|z_{n+1}| = |z_n - c(z_n)|^{-1}$  by (2). From this, we have that  $|z_1 \cdots z_{n+1}| \geq 1$  for any  $n \geq 1$ , but in fact a slightly stronger statement is true.

$$|z_1| = \left| \frac{-1}{z_0 - a_0} \right| = |z_0 - a_0|^{-1} = |z - c(z)|^{-1}$$

Suppose  $|z - c(z)| < 1$ . Then  $|z_1| > 1$ , and  $|z_1 \cdots z_{n+1}| \geq 1$  can be strengthened to  $|z_1 \cdots z_{n+1}| > 1$  for any  $n \geq 0$ .

Now suppose  $|z - c(z)| = 1$ . Then  $|z_1| = 1$ , which by the assumption of the theorem means  $|z_1 - c(z_1)| \neq 1$  and thus  $|z_1 - c(z_1)| < 1$ . Then

$$|z_2| = \left| \frac{-1}{z_1 - a_1} \right| = |z_1 - a_1|^{-1} = |z_1 - c(z_1)|^{-1} > 1.$$

Thus  $|z_1 \cdots z_{n+1}| > 1$  for any  $n \geq 1$  no matter whether  $|z - c(z)| < 1$  or not.

By Lemma 2,  $p_n - q_n z = (z_1 \cdots z_{n+1})^{-1}$  for all  $n \geq 1$  and thus  $0 \leq |p_n - q_n z| < 1$  for all  $n \geq 1$ . However  $|p_n - q_n z| = 0$  is impossible because then  $z = \frac{p_n}{q_n}$ , contradicting the condition in the theorem that  $z$  be irrational. Thus  $0 < |p_n - q_n z| < 1$  for all  $n \geq 1$ . Recall  $q_0 = 1$ . If  $q_k = 0$  for some  $k \geq 1$ , then  $|p_k - q_k z|$  would equal just  $|p_k|$ . Since  $p_k \in \mathcal{G}$ , it cannot be that  $0 < |p_k| < 1$ ; thus  $q_n \neq 0$  for all  $n \geq 0$ .

$$\begin{aligned} \left| \frac{p_n}{q_n} - z \right| &= \left| \left( \frac{p_n}{q_n} - z \right) q_n \right| |q_n|^{-1} \\ &= |p_n - q_n z| |q_n|^{-1} = |z_1 \cdots z_{n+1}|^{-1} |q_n|^{-1} \end{aligned}$$

Knowing that  $|z_1 \cdots z_{n+1}|^{-1} \rightarrow 0$  from Lemma 3 and that  $|q_n|^{-1}$  is bounded (by 1 since  $q_n \in \mathcal{G} \setminus \{0\}$  and thus  $|q_n| \geq 1$ ), we have that  $\left| \frac{p_n}{q_n} - z \right| \rightarrow 0$ . Therefore the sequence  $\{p_n/q_n\}$  converges to  $z$ .

Lastly, assume  $|q_n| \leq M$ . Then  $1 \geq |q_n|^{-1} \geq 1/M$  and

$$|p_n| \geq \frac{|p_n|}{|q_n|} \geq \frac{|p_n|}{M}.$$

If  $|p_n|$  diverges, then  $|p_n|/M$  diverges as well, but we know  $|p_n/q_n|$  converges to  $|z|$ . Thus  $|p_n|$  must converge. A converging sequence from a discrete set must be eventually constant. If  $|p_n|$  is constant for all  $n > N$ , then  $\{p_n\}_{n \geq N}$  has only a finite set of values, and since  $|q_n|$  is bounded,  $\{q_n\}_{n \geq N}$  also has only finitely many values. This means that the set  $\{p_n/q_n : n \geq N\}$  is finite. A converging sequence from a finite set must eventually equal its limit, so  $z$  must equal exactly  $p_n/q_n$  for some  $n$ . However, this contradicts the irrationality of  $z$ . For an irrational  $z$ , then, it must be that  $\{q_n\}$  is not bounded and thus that  $|q_n| \rightarrow \infty$ .  $\square$

### 3. DIAMOND ALGORITHM AND ITS PARTITION

The remainder of this paper deals exclusively with the “diamond algorithm,” which uses the fundamental set

$$\Phi_\diamond := \{x + yi : |x| + |y| \leq 1\} \quad (3)$$

and the choice function  $c_\diamond := c_{\Phi_\diamond}$  defined as described in Lemma 1. This algorithm was first described by Julius Hurwitz in 1902 [6].

The three maps

$$T(z) = z + 1, \quad U(z) = z + i, \quad S(z) = \frac{-1}{z}$$

and their inverses are the basis for various transformations related to complex continued fractions, including the piecewise continuous map  $f_\diamond : \mathbb{C} \rightarrow \mathbb{C}$  defined as

$$f_\diamond(z) = \begin{cases} Sz & \text{if } z \in \Phi_\diamond \\ T^{-1}z & \text{if } -\pi/4 \leq \arg z < \pi/4 \\ U^{-1}z & \text{if } \pi/4 \leq \arg z < 3\pi/4 \\ Tz & \text{if } 3\pi/4 \leq \arg z \text{ or } \arg z < -3\pi/4 \\ Uz & \text{if } -3\pi/4 \leq \arg z < -\pi/4. \end{cases} \quad (4)$$

The “pieces” of this piecewise definition are designed to bring any point  $z \notin \Phi_\diamond$  closer to the origin by integer translation until it enters the set  $\Phi_\diamond$  and is inverted. Figure 1 shows the regions of  $\mathbb{C}$  on which this function acts by different maps ( $\Phi_\diamond$  is shaded in the figure). A set is called *consistent* if the map  $f_\diamond$  acts on all points in the set by the same generator, meaning that the set is contained in only one of these regions.

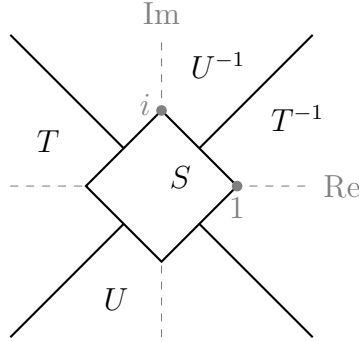


FIGURE 1. Action of  $f_\diamond$  in different regions

The natural extension map of  $f_\diamond$  is the map  $F_\diamond : \mathbb{C}^2 \setminus \Delta \rightarrow \mathbb{C}^2 \setminus \Delta$ , where  $\Delta = \{(z, w) \in \mathbb{C}^2 : z = w\}$ , given by

$$F_\diamond(z, w) = \begin{cases} (Sz, Sw) & \text{if } w \in \Phi_\diamond \\ (T^{-1}z, T^{-1}w) & \text{if } \frac{-1}{4}\pi \leq \arg w < \frac{1}{4}\pi \\ (U^{-1}z, U^{-1}w) & \text{if } \frac{1}{4}\pi \leq \arg w < \frac{3}{4}\pi \\ (Tz, Tw) & \text{if } \frac{3}{4}\pi \leq \arg w \text{ or } \arg w < \frac{-3}{4}\pi \\ (Uz, Uw) & \text{if } \frac{-3}{4}\pi \leq \arg w < \frac{-1}{4}\pi \end{cases} \quad (5)$$

and is the main object of study for the remainder of this paper.

Analogous to the “finite rectangular structure” attractor region for the real natural extension map  $F_{a,b}$  described in [10, 11], the goal is to find

an invariant set  $D_\diamond$  for the map  $F_\diamond$  that has *finite product structure*, meaning it can be expressed as a finite union of Cartesian products:

$$D_\diamond = \bigcup_{k=1}^N (Z_k \times W_k) \quad (6)$$

and each set  $W_k$  is consistent. This is accomplished by forming a finite partition of  $\mathbb{C}$  satisfying the following properties:

- (i) the set  $\Phi_\diamond$  is a union of sets from this partition;
- (ii) the image under  $f_\diamond$  of any set in the partition is a union of sets from this partition.

The natural way to do this is to look at the all images of  $\Phi_\diamond$  under  $f_\diamond$  and take a partition fine enough to describe every one of these images as a union of partition elements. A priori, there is no reason to think that such a partition will be finite or have any nice presentation, and indeed it is unclear what can be said for a generic choice function or fundamental set. In the specific case of  $f_\diamond$ , however, this partition is quite nice (see Figure 2).

Property (ii) above implies that the image under  $F_\diamond$  of any set with finite product structure must also have finite product structure.

**Lemma 5.** *Let  $W_1, \dots, W_N$  be a collection of sets such that for each  $k$  there exists a set of indices  $J_k \subset \{1, \dots, N\}$  satisfying  $f_\diamond(W_k) = \bigcup_{j \in J_k} W_j$ . Then the image under  $F_\diamond$  of any set with finite product structure must also have finite product structure.*

*Proof.* For each  $k$ , let  $h_k$  be one of the maps  $T, T^{-1}, U, U^{-1}, S$  chosen so that  $f_\diamond$  acts on  $W_k$  by the map  $h_k$ . Thus  $f_\diamond(W_k) = h_k(W_k)$  and moreover  $F_\diamond(Z_k \times W_k) = h_k(Z_k) \times h_k(W_k)$ .

Let  $A = \bigcup_{k=1}^N (Z_k \times W_k)$  be some set with finite product structure.

$$\begin{aligned} F_\diamond(A) &= F_\diamond\left(\bigcup_{1 \leq k \leq N} (Z_k \times W_k)\right) \\ &= \bigcup_{1 \leq k \leq N} F_\diamond(Z_k \times W_k) \\ &= \bigcup_{1 \leq k \leq N} h_k(Z_k) \times h_k(W_k) \\ &= \bigcup_{1 \leq k \leq N} \left(h_k(Z_k) \times \bigcup_{j \in J_k} W_j\right) \\ &= \bigcup_{1 \leq k \leq N} \bigcup_{j \in J_k} (h_k Z_k \times W_j) \end{aligned}$$



Since each  $J_k$  is finite, the double-union over  $k \in \{1, \dots, N\}$  and  $j \in J_k$  is still a finite union.  $\square$

From the eventual construction, it will be seen that equation (6) can be satisfied with  $N = 40$  sets, but a more compact description can be given using symmetry. The set  $\Phi_\diamond$  is a diamond, which has symmetry group  $\text{Dih}_4$  (the dihedral group of degree 4 and order 8), and it will be convenient to do calculations with in the quotient space  $\mathbb{C}/\text{Dih}_4$ , which is naturally identified with the “wedge-shaped” set of points  $\{w \in \mathbb{C} : 0 \leq \arg w < \pi/4\}$ , which we will denote by  $\mathbb{C}^*$ .

Define the following sets, which can be seen in Figure 2.

$$\begin{aligned} W_1 &= \{w \in \mathbb{C}^* : \text{Im } w \leq \text{Re } w - 1\} \\ W_2 &= \left\{w \in \mathbb{C}^* : \text{Im } w \geq \text{Re } w - 1, \left|w - \left(\frac{1}{2} + \frac{1}{2}i\right)\right| \geq \frac{1}{\sqrt{2}}\right\} \\ W_3 &= \left\{w \in \mathbb{C}^* : \text{Im } w \geq 1 - \text{Re } w, \left|w - \left(\frac{1}{2} + \frac{1}{2}i\right)\right| \leq \frac{1}{\sqrt{2}}\right\} \\ W_4 &= \left\{w \in \mathbb{C}^* : \text{Im } w \leq 1 - \text{Re } w, \left|w - \left(\frac{1}{2} - \frac{1}{2}i\right)\right| \geq \frac{1}{\sqrt{2}}\right\} \\ W_5 &= \left\{w \in \mathbb{C}^* : \left|w - \left(\frac{1}{2} - \frac{1}{2}i\right)\right| \leq \frac{1}{\sqrt{2}}\right\} \end{aligned} \quad (7)$$

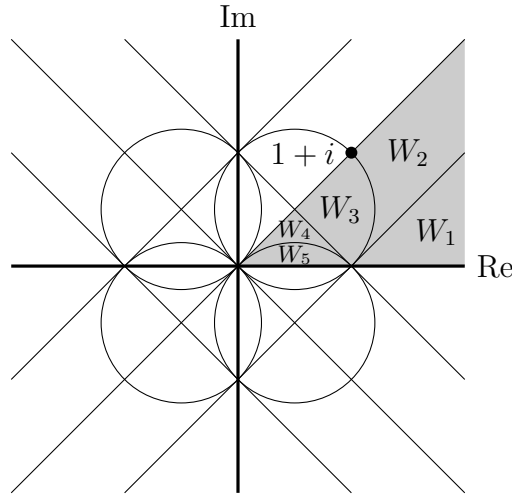


FIGURE 2. Partition of  $\mathbb{C}$  with  $\mathbb{C}^*$  in gray

The sets  $W_1, \dots, W_5$  partition the wedge  $\{w \in \mathbb{C} : 0 \leq \arg w < \pi/4\}$  (ignoring overlap on their boundaries), and thus the collection of sets  $\xi W_k$  with  $\xi \in \text{Dih}_4$  and  $k = 1, 2, 3, 4, 5$  form a partition of  $\mathbb{C}$  itself, shown in Figure 2. Since  $|\text{Dih}_4| = 8$ , there are  $8 \times 5 = 40$  total sets in this partition, but all relevant calculations can be carried out using only the 5 “standard” sets given above.

**Lemma 6.** *The partition  $\{\xi W_k : \xi \in \text{Dih}_4, 1 \leq k \leq 5\}$  satisfies the following:*

- (i) *the set  $\Phi_\diamond$  is a union of sets from this partition;*
- (ii) *the image under  $f_\diamond$  of any set in the partition is a union of sets from this partition.*

*Proof.* The set  $W_4 \cup W_5$  is the projection of  $\Phi_\diamond$  to  $\mathbb{C}/\text{Dih}_4$ , that is,

$$\Phi_\diamond = \bigcup_{\xi \in \text{Dih}_4} \xi W_4 \cup \bigcup_{\xi \in \text{Dih}_4} \xi W_5,$$

which proves (i).

To prove (ii), it is sufficient to express the images of  $W_1, \dots, W_5$  as unions of partition elements:

$$\begin{aligned} f_\diamond(W_1) &= T^{-1}W_1 = W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5 \\ f_\diamond(W_2) &= T^{-1}W_2 = \rho W_2 \cup \rho W_3 \cup \rho W_4 \\ f_\diamond(W_3) &= T^{-1}W_3 = \rho W_5 \cup \iota W_5 \cup \iota W_4 \\ f_\diamond(W_4) &= SW_4 = \eta W_2 \\ f_\diamond(W_5) &= SW_5 = \eta W_1, \end{aligned} \tag{8}$$

where  $\rho : w \mapsto i\bar{w}$  (reflection across  $\text{Im } w = \text{Re } w$ ),  $\iota : w \mapsto iw$  (counterclockwise rotation by  $90^\circ$ ), and  $\eta : w \mapsto -\bar{w}$  (reflection across the  $\text{Im}$ -axis) are all elements of  $\text{Dih}_4$ .  $\square$

#### 4. BIJECTIVITY DOMAIN

We can now use the sets  $W_k$  in the partition to describe a bijectivity domain for  $F_\diamond$ .

**Theorem 7.** *There exists a set  $D_\diamond \subset \mathbb{C}^2 \setminus \Delta$  such that*

- (i)  *$D_\diamond$  is a bijectivity domain, meaning that  $F_\diamond(D_\diamond) = D_\diamond$  and the map  $F_\diamond : D_\diamond \rightarrow D_\diamond$  is bijective except on parts of the boundary of  $D_\diamond$ ;*
- (ii)  *$D_\diamond$  has finite product structure: there exists a finite collections of sets  $Z_1, W_1, \dots, Z_N, W_N \subset \mathbb{C}$  such that*

$$D_\diamond = \bigcup_{k=1}^N (Z_k \times W_k); \tag{6}$$

- (iii) *each set  $Z_k$  and  $W_k$  in (6) can be given explicitly, and each is a connected set whose boundary consists of straight lines (infinite or segments) and arcs of circles.*

**Remark.** Ideally,  $D_\diamond$  should actually be an attractor region, which would require that the orbit of almost any point  $(z, w)$  in  $\mathbb{C}^2 \setminus \Delta$  enters  $D_\diamond$  in finite time or at least asymptotically. This has yet to be proved or disproved. Section 5 describes a set  $\Psi_\diamond \supset D_\diamond$  with this property.

The proof of Theorem 7 consists of descriptions of the sets  $Z_k$  and  $W_k$  satisfying (iii), defining the set  $D_\diamond$  as in (ii), and calculations to prove (i). This information is presented below along with the method used to construct and determine these sets; see the left half of Figure 3 for a visualization of  $D_\diamond$ .

The sets  $Z_1, \dots, Z_5$  for equation (6) are the following:

$$\begin{aligned} Z_1 &= \{ z \in \mathbb{C} : \operatorname{Re} z \leq \tfrac{1}{2}, |\operatorname{Im} z| \leq \tfrac{1}{2}, |z - 1| \geq 1 \} \\ Z_2 &= \{ z \in \mathbb{C} : \operatorname{Re} z \leq \tfrac{1}{2}, \operatorname{Im} z \leq \tfrac{1}{2}, |z - 1| \geq 1 \} \\ Z_3 &= \{ z \in \mathbb{C} : \operatorname{Re} z \leq \tfrac{1}{2}, \operatorname{Im} z \leq \tfrac{1}{2}, |z| \geq 1 \} \\ Z_4 &= \{ z \in \mathbb{C} : \operatorname{Re} z \leq \tfrac{1}{2}, |z| \geq 1, |z - i| \geq 1 \} \\ Z_5 &= \{ z \in \mathbb{C} : \operatorname{Re} z \leq \tfrac{1}{2}, |z| \geq 1, |z - i| \geq 1, |z + i| \geq 1 \} \end{aligned} \quad (9)$$

Using  $W_1, \dots, W_5$  as defined in (7) and  $Z_1, \dots, Z_5$  as defined in (9), we define the set  $D_\diamond$  as

$$D_\diamond = \bigcup_{k=1}^5 \bigcup_{\xi \in \operatorname{Dih}_4} \xi(Z_k \times W_k), \quad (10)$$

where  $\operatorname{Dih}_4$  acts on  $\mathbb{C}^2$  by  $\xi(z, w) = (\xi z, \xi w)$ . Equation (10) is a restatement of (6) with  $Z_6, \dots, Z_{40}$  each being an image of one of  $Z_1, \dots, Z_5$  under an element of  $\operatorname{Dih}_4$  and likewise for  $W_6, \dots, W_{40}$ . The set  $D_\diamond$  can most easily be visualized using the five products on the left of Figure 3. **Note that each product  $Z \times W$  shown in Figure 3 actually represents  $\bigcup_{\xi \in \operatorname{Dih}_4} \xi(Z \times W)$ .**

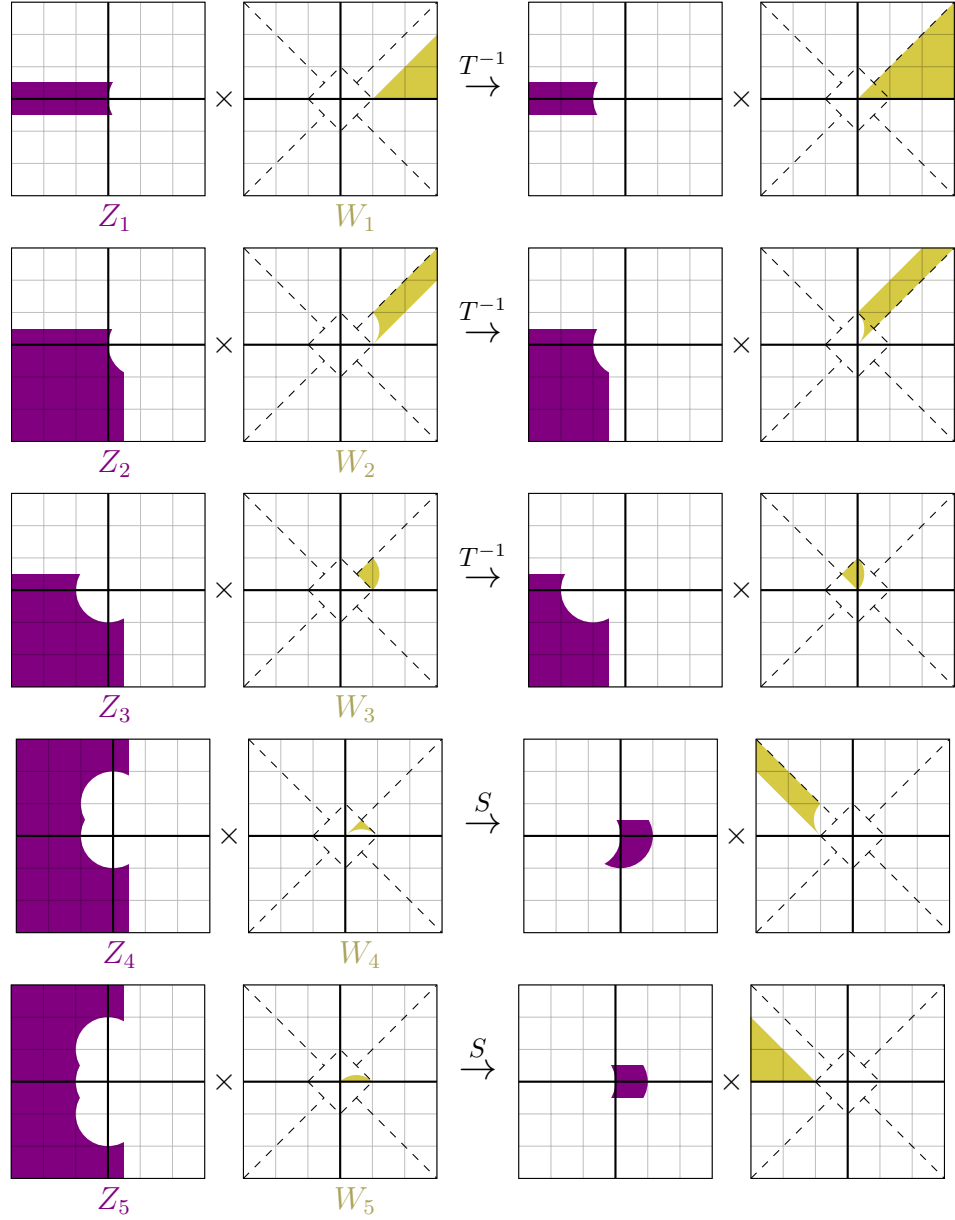
As proved in Lemma 5, property (ii) of Lemma 6 implies that  $F_\diamond(D_\diamond)$  must have finite product structure with exact same sets  $W_k$ . Thus there must exist sets  $\hat{Z}_k$  such that

$$F_\diamond(D_\diamond) = \bigcup_{k=1}^{40} (\hat{Z}_k \times W_k) = \bigcup_{k=1}^5 \bigcup_{\xi \in \operatorname{Dih}_4} \xi(\hat{Z}_k \times W_k).$$

Now proving that  $F_\diamond(D_\diamond) = D_\diamond$  is equivalent to proving that  $\hat{Z}_k = Z_k$  for  $k = 1, \dots, 5$ .

Notice that  $\hat{Z}_k \times W_k = F_\diamond(D_\diamond) \cap (\mathbb{C} \times W_k)$ , so we can describe each  $\hat{Z}_k$  as

$$\hat{Z}_k = \{ z \in \mathbb{C} : \exists w \in W_k \text{ such that } (z, w) \in F_\diamond(D_\diamond) \}. \quad (11)$$

FIGURE 3. The products  $Z_k \times W_k$  and their images under  $F_\diamond$ 

Let us calculate  $\hat{Z}_1$  explicitly here. Based on equation (8) and seen in the top and bottom rows of Figure 3, the images  $f_\diamond(W_1)$  and  $f_\diamond(W_5)$  contain some set  $\xi W_1, \xi \in \text{Dih}_4$ . Therefore finding the expression for  $\hat{Z}_1$  based on (11) will involve the images

$$\begin{aligned} F_\diamond(Z_1 \times W_1) &= T^{-1}Z_1 \times T^{-1}W_1 \\ &= T^{-1}Z_1 \times (W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5) \end{aligned}$$

and

$$F_\diamond(Z_5 \times W_5) = SZ_5 \times SW_5 = SZ_5 \times \eta W_1.$$

Since  $W_1 \subset T^{-1}W_1$ , we have that

$$T^{-1}Z_1 \times W_1 \subset F_\diamond(Z_1 \times W_1)$$

and therefore

$$T^{-1}Z_1 \subset \hat{Z}_1. \quad (12)$$

Based on the symmetry of  $D_\diamond$ , the sets  $\xi(SZ_5 \times \eta W_1)$  for all  $\xi \in \text{Dih}_4$  are also contained in  $F_\diamond(D_\diamond)$ . Using  $\xi = (\eta)^{-1} = \eta$  gives that

$$\eta^{-1}(SZ_5 \times \eta W_1) = \eta SZ_5 \times W_1 \subset F_\diamond(Z_5 \times Z_5)$$

and therefore

$$\eta SZ_5 \subset \hat{Z}_1. \quad (13)$$

Since the images of  $W_2$ ,  $W_3$ , and  $W_4$  do not contain any  $\xi W_1$ , there are no more subsets of  $\hat{Z}_1$  to find. Equations (12) and (13) therefore fully describe  $\hat{Z}_1$ , which can now be given as

$$\hat{Z}_1 = T^{-1}Z_1 \cup \eta SZ_5.$$

Figure 4 shows that  $T^{-1}Z_1 \cup \eta SZ_5$  is exactly equal to  $Z_1$ . Thus  $\hat{Z}_1 = Z_1$  as desired.

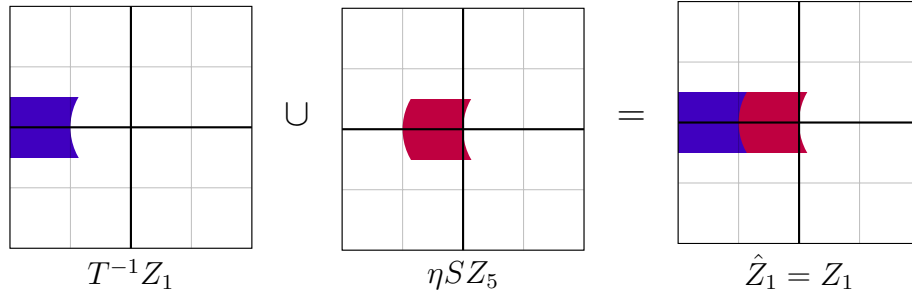


FIGURE 4.  $\hat{Z}_1 = Z_1$  as a union of two pieces

To calculate  $\hat{Z}_2$  explicitly, we see<sup>2</sup> in equation (8) that the images

$$f_\diamond(W_1) = T^{-1}W_1, \quad f_\diamond(W_2) = T^{-1}W_2, \quad f_\diamond(W_4) = SW_4$$

all contain some set  $\xi W_2$  and construct  $\hat{Z}_2$  by piecing together the appropriate symmetric copies of  $T^{-1}Z_1$ ,  $T^{-1}Z_2$ , and  $SZ_4$ . This gives

$$\hat{Z}_2 = T^{-1}Z_1 \cup \rho T^{-1}Z_2 \cup \eta SZ_4,$$

and indeed  $\hat{Z}_2 = Z_2$ . Similarly, one can show

$$\begin{aligned} \hat{Z}_3 &= T^{-1}Z_1 \cup \rho T^{-1}Z_2 &= Z_3, \\ \hat{Z}_4 &= T^{-1}Z_1 \cup \rho T^{-1}Z_2 \cup \iota^3 T^{-1}Z_3 &= Z_4, \\ \hat{Z}_5 &= T^{-1}Z_1 \cup \iota^3 T^{-1}Z_3 \cup \rho T^{-1}Z_3 &= Z_5. \end{aligned}$$

Note the expression for  $\hat{Z}_5$  correctly includes two different symmetric copies of  $T^{-1}Z_3$  because the expression for  $T^{-1}W_3$  in (8) contains two symmetric copies of  $W_5$ . Here  $\iota^3 : z \mapsto -iz$  and  $\rho : z \mapsto i\bar{z}$ .

<sup>2</sup>This can be literally seen in the first, second, and fourth rows of Figure 3.

The expressions above show that  $\hat{Z}_k = Z_k$  for  $k = 1, \dots, 5$ , which proves that  $F_\diamond(D_\diamond) = D_\diamond$ . Moreover, the unions in these expressions are disjoint except for the boundaries of the pieces, so the map  $F_\diamond$  is essentially bijective on  $D_\diamond$ . This completes the proof of Theorem 7.

## 5. TRAPPING REGION

If the orbit of a point  $(z, w) \in \mathbb{C}^2 \setminus \Delta$  under  $F_\diamond$  enters the set  $D_\diamond$  it will never leave, but do all orbits enter  $D_\diamond$ ? Unfortunately, this is not currently known, but we now exhibit a set  $\Psi_\diamond \supset D_\diamond$  that does “trap points.”

**Theorem 8.** *Let  $W_1, \dots, W_5$  be the sets defined in (7). There exist sets  $A_1, \dots, A_5 \subset \mathbb{C}$  such that the set*

$$\Psi_\diamond = \bigcup_{k=1}^5 \bigcup_{\xi \in \text{Dih}_4} \xi(A_k \times W_k)$$

*has the following properties:*

- (i) *for each  $(z, w) \in \mathbb{C}^2 \setminus \Delta$  with  $w$  irrational, there exists an integer  $N \geq 0$  such that  $F_\diamond^N(z, w) \in \Psi_\diamond$ ;*
- (ii)  $F_\diamond(\Psi_\diamond) \subset \Psi_\diamond$ ;
- (iii) *each  $A_k$  can be explicitly described as a countable union of unit disks and half-planes;*
- (iv)  $D_\diamond \subset \Psi_\diamond$ .

Properties (i) and (ii) together mean that  $\Psi_\diamond$  is a *trapping region*. The exclusion of rational  $w$ -values in property (i) is because those points have finite orbits under  $F_\diamond$ . Also note that property (iv) is equivalent to  $Z_k \subset A_k$  for  $k = 1, \dots, 5$ .

Before attempting to describe the sets  $A_k$ , we first look at a set  $V_\diamond$  satisfying only (i).

**Lemma 9.** *Define*

$$V_\diamond := \overline{\mathbb{D}} \times S(\Phi_\diamond). \tag{14}$$

*For all  $(z, w)$  with  $w$  irrational, there exists an integer  $N \geq 0$  such that  $F_\diamond^N(z, w) \in V_\diamond$ .*

*Proof.* Since  $w$  is irrational, its continued fraction expansion is infinite. Let

$$w = [n_0 + m_0i, n_1 + m_1i, \dots] = (n_0 + m_0i) - \frac{1}{(n_1 + m_1i) - \frac{1}{\ddots}}$$

be the continued fraction for  $w$  generated by the diamond algorithm. That is,

$$w_0 = w, \quad n_k + m_ki = c_\diamond(w_k), \quad w_k = \frac{-1}{w_k - (n_k + m_ki)},$$

where  $c_\diamond := c_{\Phi_\diamond}$  is the diamond choice defined as described in Lemma 1. Equivalently,  $c_\diamond(w_k) = w_k - f_\diamond^j(w_k)$  where  $j$  is the smallest natural number for which  $f_\diamond^j(w_k) \in \Phi_\diamond$ .

In terms of the functions  $T, T^{-1}, U, U^{-1}$ , and  $S$ , we can write

$$w_{k+1} = SU^{-m_k}T^{-n_k} \dots SU^{-m_1}T^{-n_1}SU^{-m_0}T^{-n_0}w.$$

Construct the sequence  $\{(z_k, w_k)\}$  with  $w_k$  as above and

$$z_{k+1} := SU^{-m_k}T^{-n_k} \dots SU^{-m_1}T^{-n_1}SU^{-m_0}T^{-n_0}z.$$

Note that the points  $\{(z_k, w_k)\}$  are all in the orbit of  $(z_0, w_0) = (z, w)$  under  $F_\diamond$ , specifically those points for which  $F_\diamond$  acted by  $S(z, w) = (-1/z, -1/w)$  in its last iteration. Since inversion was the last operation applied, it must be that  $S^{-1}w_{k+1} \in \Phi_\diamond$  and thus  $w_{k+1} \in S(\Phi_\diamond)$ .

Let  $\frac{p_k}{q_k}$  be the  $k^{\text{th}}$  convergents for  $w$ . Then

$$\begin{aligned} z &= T^{n_0}U^{m_0}ST^{n_1}U^{m_1}S \dots T^{n_k}U^{m_k}S(z_{k+1}) = \frac{p_k z_{k+1} - p_{k-1}}{q_k z_{k+1} - q_{k-1}}, \\ w &= T^{n_0}U^{m_0}ST^{n_1}U^{m_1}S \dots T^{n_k}U^{m_k}S(w_{k+1}) = \frac{p_k w_{k+1} - p_{k-1}}{q_k w_{k+1} - q_{k-1}}. \end{aligned}$$

Hence

$$z_{k+1} = \frac{q_{k-1}z - p_{k-1}}{q_k z - p_k} = \frac{q_{k-1}}{q_k} + \frac{1}{q_k^2(\frac{p_k}{q_k} - z)}.$$

Setting  $\epsilon_k := q_k^{-2}(\frac{p_k}{q_k} - z)^{-1}$ , we have

$$\begin{aligned} z_{k+1} &= \frac{q_{k-1}}{q_k} + \epsilon_k \\ |z_{k+1}| &\leq \frac{|q_{k-1}|}{|q_k|} + |\epsilon_k|. \end{aligned}$$

Since  $\frac{p_k}{q_k} \rightarrow w \neq z$  by Theorem 4, we have  $|\epsilon_k| < \frac{1}{|q_k|^2}$ . Thus

$$|z_{k+1}| < \frac{|q_{k-1}|}{|q_k|} + \frac{1}{|q_k|^2} = \frac{|q_{k-1}||q_k| + 1}{|q_k|^2}.$$

Since  $|q_k| \rightarrow \infty$  by Theorem 4, there must be an infinite subsequence  $|q_{k_j}|$  that is strictly increasing. Thus there are infinitely many  $k$  for

which  $\frac{|q_{k-1}|}{|q_k|} < 1$ . Thus there exists  $k$  such that  $|z_k| \leq 1$ , equivalently,  $z_k \in \overline{\mathbb{D}}$ , and we already know that  $w_k \in S(\Phi_\diamond)$ . This means precisely that  $(z_k, w_k) \in V_\diamond = \overline{\mathbb{D}} \times S(\Phi_\diamond)$ .  $\square$

Although the orbit of every irrational point enters  $V_\diamond$  by Lemma 9, orbits will quickly leave  $V_\diamond$  as well. One way to create a trapping region is to take unions of the images of  $V_\diamond$ . That is, the set

$$\Psi_\diamond := \bigcup_{n \geq 0} F_\diamond^n(V_\diamond) \quad (15)$$

will be a trapping region by construction.

Equation (15) actually serves as the definition of  $\Psi_\diamond$ , and it remains to show that this set has all the properties claimed in Theorem 8. The existence of some sets  $A_k$  satisfying  $\Psi_\diamond = \bigcup_{k=1}^5 A_k \times W_k \bmod \text{Dih}_4$  follows immediately from the finite product structure of  $V_\diamond$ ; the finite trapping property (i) follows from  $V_\diamond \subset \Psi_\diamond$ ; and property (ii),  $F_\diamond(\Psi_\diamond) \subset \Psi_\diamond$ , holds by construction. Thus it is really the explicit description of the sets, property (iii), that is noteworthy.

*Proof of (iii).* Modulo  $\text{Dih}_4$ , we have  $S(\Phi_\diamond) = W_1 \cup W_2$ . That is, the actual equality is  $S(\Phi_\diamond) = \bigcup_{\xi \in \text{Dih}_4} \xi(W_1 \cup W_2)$ , but any calculations involving  $S(\Phi_\diamond)$  can be done using only  $W_1$  and  $W_2$ . The remaining calculations are done mod  $\text{Dih}_4$ ; we start with

$$V_\diamond = \overline{\mathbb{D}} \times W_1 \cup \overline{\mathbb{D}} \times W_2.$$

Then

$$F_\diamond(\mathbb{D} \times W_1) = \overline{B}(-1) \times (W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5)$$

and

$$F_\diamond(\mathbb{D} \times W_2) = \overline{B}(-1) \times (\rho W_2 \cup \rho W_3 \cup \rho W_4),$$

where  $\overline{B}(c)$  is the unit ball in  $\mathbb{C}$  with center  $c$  and  $\rho \in \text{Dih}_4$  is the reflection  $\rho : w \mapsto i\overline{w}$ . Since  $\rho^{-1}\overline{B}(-1) = \rho\overline{B}(-1) = \overline{B}(-i)$ , we have

$$F_\diamond(V_\diamond) = \overline{B}(-1) \times W_1 \cup [\overline{B}(-1) \cup \overline{B}(-i)] \times [W_2 \cup W_3 \cup W_4].$$

The image  $F_\diamond^2(V_\diamond)$  will include half-planes because the image of the disk  $\overline{B}(-1)$  under  $S : z \mapsto -1/z$  is the half-plane  $\{z : \text{Re } z > \frac{1}{2}\}$ . Actually,  $S(W_4) = \eta W_2$  and  $S(W_5) = \eta W_1$  with  $\eta : (x, y) \mapsto (-x, y)$ , so we end up with the half-plane  $\{z : \text{Re } z < \frac{-1}{2}\}$  after flipping the sets back around (the product  $\{z : \text{Re } z > \frac{1}{2}\} \times \eta W_2$  is equivalent mod  $\text{Dih}_4$  to



$\{z : \operatorname{Re} z < \frac{-1}{2}\} \times W_2$ , and likewise for a product with  $\eta W_1$ ).

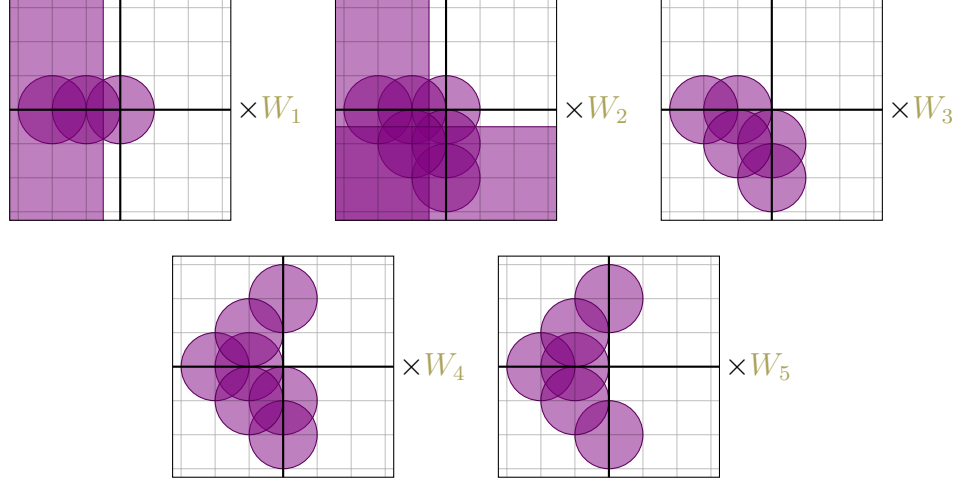
$$\begin{aligned}
 F_\diamond(\overline{B}(-1) \times W_1) &= \overline{B}(-2) \times (W_1 \cup W_2 \cup W_3 \cup W_4 \cup W_5) \\
 F_\diamond([\overline{B}(-1) \cup \overline{B}(-i)] \times W_2) &= [\overline{B}(-2) \cup \overline{B}(-1-i)] \times (\rho W_2 \cup \rho W_3 \cup \rho W_4) \\
 &= [\overline{B}(-2i) \cup \overline{B}(-1-i)] \times (W_2 \cup W_3 \cup W_4) \\
 F_\diamond([\overline{B}(-1) \cup \overline{B}(-i)] \times W_3) &= [\overline{B}(-2) \cup \overline{B}(-1-i)] \times (\rho W_5 \cup iW_5 \cup iW_4) \\
 &= [\overline{B}(-2) \cup \overline{B}(-1-i)] \times \rho W_5 \\
 &\quad \cup [\overline{B}(-2) \cup \overline{B}(-1-i)] \times i(W_5 \cup W_4) \\
 &= [\overline{B}(-2i) \cup \overline{B}(-1-i)] \times W_5 \\
 &\quad \cup [\overline{B}(2i) \cup \overline{B}(-1+i)] \times (W_5 \cup W_4) \\
 F_\diamond([\overline{B}(-1) \cup \overline{B}(-i)] \times W_4) &= [\{z : \operatorname{Re} z > \frac{1}{2}\} \cup \{z : \operatorname{Im} z < \frac{-1}{2}\}] \times \eta W_2 \\
 &= [\{z : \operatorname{Re} z < \frac{-1}{2}\} \cup \{z : \operatorname{Im} z < \frac{-1}{2}\}] \times W_2 \\
 F_\diamond(\overline{B}(-1) \times W_5) &= \{z : \operatorname{Re} z > \frac{1}{2}\} \times \eta W_1 \\
 &= \{z : \operatorname{Re} z < \frac{-1}{2}\} \times W_1.
 \end{aligned}$$

Now we can collect “like terms” (products with the same  $W_k$ ) to get

$$\begin{aligned}
 F_\diamond^2(V_\diamond) &= [\overline{B}(-2) \cup \{z : \operatorname{Re} z < \frac{-1}{2}\}] \times W_1 \\
 &\quad \cup [\overline{B}(-2) \cup \overline{B}(-2i) \cup \overline{B}(-1-i) \\
 &\quad \quad \cup \{z : \operatorname{Re} z < \frac{-1}{2}\} \cup \{z : \operatorname{Im} z < \frac{-1}{2}\}] \times W_2 \\
 &\quad \cup [\overline{B}(-2) \cup \overline{B}(-2i) \cup \overline{B}(-1-i)] \times W_3 \\
 &\quad \cup [\overline{B}(-2) \cup \overline{B}(-2i) \cup \overline{B}(-1-i) \cup \overline{B}(2i) \cup \overline{B}(-1+i)] \times W_4 \\
 &\quad \cup [\overline{B}(-2) \cup \overline{B}(-2i) \cup \overline{B}(-1-i) \cup \overline{B}(2i) \cup \overline{B}(-1+i)] \times W_5.
 \end{aligned}$$

Using inclusions such as  $\overline{B}(-2) \subset \{z : \operatorname{Re} z < \frac{-1}{2}\}$ , the expression of  $F_\diamond^2(V_\diamond)$  as a union of products can be reduced somewhat, and the expression of  $V_\diamond \cup F_\diamond(V_\diamond) \cup F_\diamond^2(V_\diamond)$  can be reduced significantly.

**Remark.** Geometrically, the “outer layer” of disks (furthest centers from origin in the 1-norm) in the products with  $W_3, W_4, W_5$  in Figure 5 continue to move away from the origin with higher iterations of  $F_\diamond$ .

FIGURE 5. Products in the set  $V_\diamond \cup F_\diamond(V_\diamond) \cup F_\diamond^2(V_\diamond)$ 

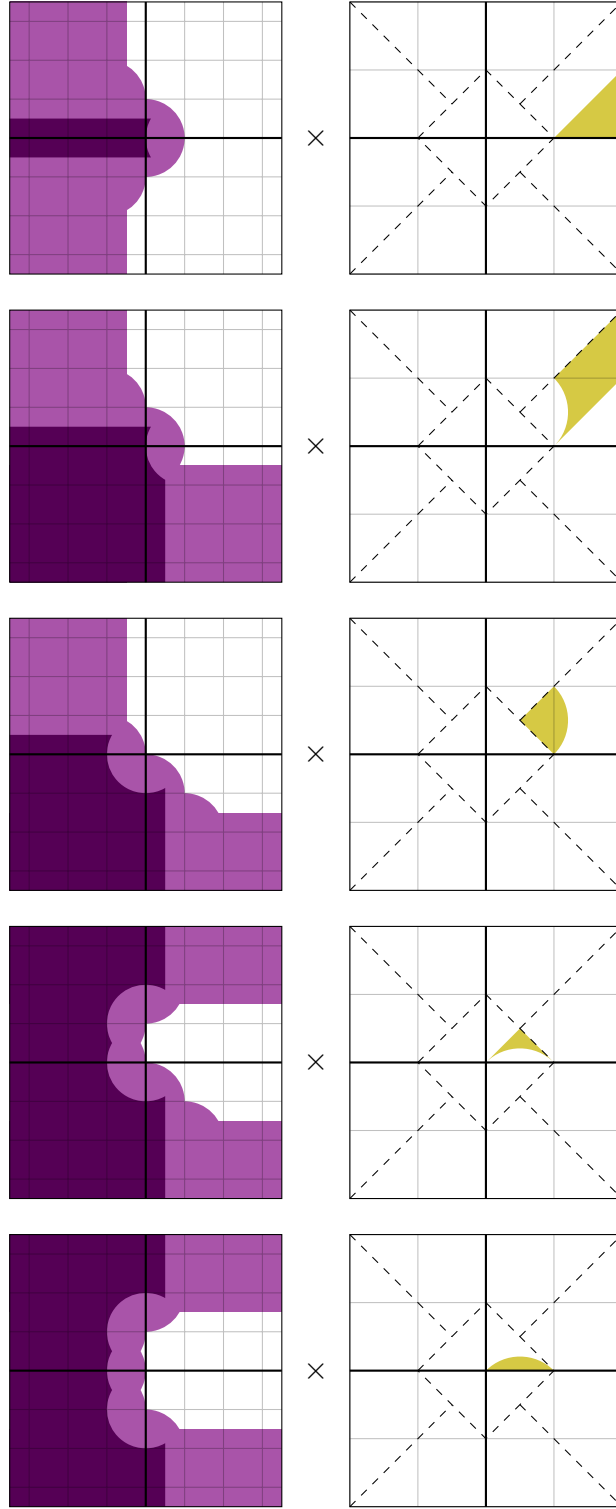
This process continues with  $F_\diamond^3(V_\diamond)$ ,  $F_\diamond^4(V_\diamond)$ , and  $F_\diamond^5(V_\diamond)$ , at which one finds that  $\bigcup_{n=0}^5 F_\diamond^n(V_\diamond) = \bigcup_{n=0}^4 F_\diamond^n(V_\diamond)$ , so we can stop iterating and terminate with  $\Psi_\diamond = \bigcup_{n=0}^4 F_\diamond^n(V_\diamond)$ .

A graphical depiction of  $\Psi_\diamond$  can be found in Figure 6. In formulas, the sets are as follows:

$$\begin{aligned}
 A_1 &= \overline{B}(0) \cup \overline{B}(-1+i) \cup \overline{B}(-1-i) \cup \left\{ z : \operatorname{Re} z \leq \frac{-1}{2} \right\} \\
 A_2 &= \overline{B}(0) \cup \overline{B}(-1+i) \cup \left\{ z : \operatorname{Re} z \leq \frac{-1}{2} \right\} \cup \left\{ z : \operatorname{Im} z \leq \frac{-1}{2} \right\} \\
 A_3 &= \overline{B}(-1) \cup \overline{B}(-i) \cup \overline{B}(1-2i) \\
 &\quad \cup \left\{ z : \operatorname{Re} z \leq \frac{-1}{2} \right\} \cup \left\{ z : \operatorname{Im} z \leq \frac{-3}{2} \right\} \\
 A_4 &= \overline{B}(2i) \cup \overline{B}(-1+i) \cup \overline{B}(-1) \cup \overline{B}(-i) \cup \overline{B}(1-2i) \\
 &\quad \cup \left\{ z : \operatorname{Re} z \leq \frac{-1}{2} \right\} \cup \left\{ z : |\operatorname{Im} z| \geq \frac{3}{2} \right\} \\
 A_5 &= \overline{B}(2i) \cup \overline{B}(-2i) \cup \overline{B}(-1+i) \cup \overline{B}(-1-i) \cup \overline{B}(-1) \\
 &\quad \cup \left\{ z : \operatorname{Re} z \leq \frac{-1}{2} \right\} \cup \left\{ z : |\operatorname{Im} z| \geq \frac{3}{2} \right\}
 \end{aligned} \tag{16}$$

In equation (16), each  $A_k$  is a countable union of unit disks and half-planes, so these descriptions agree with property (iii) of Theorem 8.  $\square$

Figure 6 shows the set  $\Psi_\diamond$  and the bijectivity domain  $D_\diamond$  calculated previously. As usual, the set  $\Psi_\diamond$  is really  $\bigcup_{k=1}^5 \bigcup_{\xi \in \operatorname{Dih}_4} (\xi A_k \times \xi W_k)$  even though only  $A_1 \times W_1, \dots, A_5 \times W_5$  are shown in the figure. From these pictures it is clear that  $Z_k \subset A_k$  for all  $k$  (this can also be verified using equations (9) and (16)), which implies property (iv),  $D_\diamond \subset \Psi_\diamond$ . This completes the proof of Theorem 8 in its entirety.

FIGURE 6. Trapping set  $\Psi_\diamond$  (light) and bijectivity domain  $D_\diamond$  (dark).

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